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SECOND-ORDER FREDHOLM EQUATIONS FOR THE FIRST BOUNDARY-VALUE PROBLEM IN THE TWO-DIMENSIONAL ANISOTROPIC THEORY OF ELASTICITY

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A complete potential theory is constructed for the first boundary-value problem in the two-dimensional anisotropic theory of elasticity (the force vector is specified on the boundary) in a bounded domain on a plane with a Lyapunov boundary.

Key words: *theory of elasticity, Fredholm equation.*

Introduction. The method of integral equations has been actively used in the theory of elasticity (see, for example, [1–4]). In this case, boundary-value problems are reduced, as a rule, to systems of singular integral equations. If the basic boundary-value problem has a zero index, its corresponding system of equations can be regularized; i.e., it can be reduced to a system of second-order Fredholm equations. The question arises: can it be at once reduced to a regular system of equations? In the three-dimensional theory of elasticity there is the so-called Weil antenna potential [5], which is used to obtain a system of regular integral equations. As noted in [4], it corresponds to an elastic solution obtained by superposition of the solutions for half-space loaded over the surface by a point force (the Boussinesq solution). In the two-dimensional theory of elasticity for isotropic materials there is a similarly constructed solution [6] of the static problem for external forces specified on the boundary. In the theory of elasticity, Sherman [7] proposed a system of second-order Fredholm equations that allows a solution of the boundary-value elasticity problem for the force vector specified on the boundary. Later the author of the present paper gave a simplified derivation of it [8]. This system of equations has a significant drawback: instead of forces, its right side contains their integrals along the boundaries. Therefore, it is desirable to have a similar system of equations in which the right side is the force vector specified on the boundary. Such a system of equations is constructed in the present paper. The proposed approach consists of constructing an analog of the Weil antenna potential for anisotropic media. For isotropic media, this leads to the system of equations written in complex form in [6]. We note that this system of equations does not coincide with the system obtained by the conventional approach using the fundamental solution of the basic system of differential equations since that approach leads to a system of singular integral equations. To derive this system, one does not need to know the fundamental solution of the elasticity equations and perform cumbersome calculations. It suffices to know that the boundary-value problem satisfies the Lopatinskii condition [9]. There are several rather complex formulations of this condition; we only note that it amounts to the existence of a single solution (decaying at infinity) of the elliptic boundary-value problem in any half-plane tangent to the boundary of the domain and implies that a certain determinant related to the boundary-value problem is different from zero. The system of equations of the two-dimensional anisotropic theory of elasticity has simple complex characteristics, which considerably simplifies its construction. This system of equations is adequate to the boundary-value problem and allows one to use minimum assumptions on the smoothness of the boundary and boundary data in the Hölder classes of functions. It can also be used in the case where the cores of integral operators lose the Fredholm property. In this case, passage to the limit for

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an isotropic material does not involve any difficulties. A similar system of equations was constructed in [10] but this paper has a number of significant drawbacks; namely, it is not proved that the cores of the integral operators appearing in the system have a weak singularity or are continuous (therefore, the equations cannot be considered Fredholm equations), and accurate assumptions on the smoothness of the boundary of the domain are not indicated. It should be noted that in the literature there is no theorem on the solvability of the examined boundary-value problem under these assumptions even for isotropic materials.

1. Let us carry out a formal construction of this system of equations under the assumption of correctness of the calculations performed, which will be justified below. Using the notation of [11], the stresses σ_{ij} $(i,j=1,2)$ are expressed in terms of the derivatives of the analytical functions $\Phi_k(z_k)$ $(z_k = x_1 + \mu_k x_2)$ as follows:

$$\sigma_{11} = \mathrm{Re}\,[\mu_1^2 \Phi_1'(z_1) + \mu_2^2 \Phi_2'(z_2)], \qquad \sigma_{22} = \mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2)],$$

$$\sigma_{12} = -\,\mathrm{Re}\,[\mu_1 \Phi_1'(z_1) + \mu_2 \Phi_2'(z_2)].$$

In this case, the displacement vector is determined with accuracy up to a rigid displacement and is given by

$$u_1 = \mathrm{Re}\,[p_1 \Phi_1(z_1) + p_2 \Phi_2(z_2)] + \alpha x_2 + \beta_1,$$

$$u_2 = \mathrm{Re}\,[q_1 \Phi_1(z_1) + q_2 \Phi_2(z_2)] - \alpha x_1 + \beta_2.$$

Here μ_k $(k=1,2)$ are complex parameters of the anisotropic material, α and β_k $(k=1,2)$ are real constants, $p_k = a_{11}\mu_k^2 + a_{12} - a_{16}\mu_k$ and $q_k = a_{12}\mu_k + a_{22}\mu_k^{-1} - a_{26}$ $(k=1,2)$. The coefficients a_{ij} $(i,j=1,2,6)$ are determined from Hooke's law

$$e_{11} = a_{11}\sigma_{11} + a_{12}\sigma_{22} + a_{16}\sigma_{12}, \qquad e_{22} = a_{12}\sigma_{11} + a_{22}\sigma_{22} + a_{26}\sigma_{12},$$

$$2e_{12} = a_{16}\sigma_{11} + a_{26}\sigma_{22} + a_{66}\sigma_{12}, \qquad e_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i,j = 1,2,$$

where e_{ij} $(i,j=1,2)$ are strains. It is assumed that the matrix of the elastic constants is positively determined. We denote by $\boldsymbol{n} = (n_1, n_2)$ $[n_1 = -x_2'(s)$ and $n_2 = x_1'(s)]$ the vector of the internal normal to the straightened Jordan boundary ∂Q of length L of a simply connected bounded domain Q on a plane. In addition, $x_1(s), x_2(s) \in C^{1,\lambda}(0,L)$, $0 < \lambda < 1$, i.e., the functions $x_1(s)$ and $x_2(s)$ that specify the shape of the boundary are continuous and are continuously differentiable, and the normal vector satisfies Lyapunov condition. The origin is inside the domain.

The boundary conditions in terms of forces are written as

$$\sigma_{11}n_1 + \sigma_{12}n_2\Big|_{\partial Q} = g_1(s), \qquad \sigma_{12}n_1 + \sigma_{22}n_2\Big|_{\partial Q} = g_2(s), \tag{1}$$

where s is an arc length reckoned counterclockwise from a certain fixed point on the boundary. We set $t_k(s) = x_1(s) + \mu_k x_2(s)$ $(k=1,2)$. Boundary conditions (1) are then written as

$$\begin{gathered}\mathrm{Re}\,\{-\mu_1 t_1'(s_0)\Phi_1'(t_1(s_0)) - \mu_2 t_2'(s_0)\Phi_2'(t_2(s_0))\} = g_1(s_0), \\ \mathrm{Re}\,\{t_1'(s_0)\Phi_1'(t_1(s_0)) + t_2'(s_0)\Phi_2'(t_2(s_0))\} = g_2(s_0).\end{gathered} \tag{2}$$

Here $t_k(s_0) = x_1(s_0) + \mu_k x_2(s_0)$ $(k=1,2)$; the prime denotes the derivative with respect to s. We write the functions $\Phi_1'(z_1)$ and $\Phi_2'(z_2)$ as Cauchy type integrals:

$$\Phi_k'(z_k) = \frac{1}{\pi i}\int\limits_{\partial Q} \frac{b_k(s)(t_k(s))^{-1}\,ds}{t_k - z_k}, \qquad k = 1,2.$$

The densities $b_k(s)$ $(k=1,2)$ appearing in these integrals are complex and are determined by solving the simple system of equations

$$-\mu_1 b_1 - \mu_2 b_2 = f_1(s), \qquad b_1 + b_2 = f_2(s). \tag{3}$$

In (3), the functions $f_k(s)$ $(k=1,2)$ are real. Let us clarify how system (3) appears. We consider the particular case — the solution of the initial boundary-value problem in the half-plane $x_2 > 0$. We write the functions appearing in the solution as Cauchy integrals and attempt to satisfy boundary conditions (1). Then, the densities appearing in these integrals satisfy relation (3). For the half-plane that is tangent to the domain at any point of the boundary (in a local rectangular coordinate system attached to this point), the solution of this problem is also system (3). Solving it, we obtain

$$\Phi_1'(z_1) = -\frac{1}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(f_1+\mu_2 f_2)[t_1'(s)]^{-1}\,dt_1}{t_1-z_1},$$

$$\Phi_2'(z_2) = \frac{1}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(f_1+\mu_1 f_2)[t_2'(s)]^{-1}\,dt_2}{t_2-z_2}.$$

For the Cauchy integral, the Sokhotskii formulas hold [6]:

$$\lim_{z_j\to t_j}\frac{1}{\pi i}\int\limits_{\partial Q}\varphi(s)\,\frac{dt_j}{t_j-z_j} = \varphi(s_0)+\frac{1}{\pi i}\int\limits_{\partial Q}\varphi(s)\,\frac{dt_j}{t_j-t_{j0}}, \qquad z\in Q_i,$$

$$\lim_{z_j\to t_j}\frac{1}{\pi i}\int\limits_{\partial Q}\varphi(s)\,\frac{dt_j}{t_j-z_j} = -\varphi(s)+\frac{1}{\pi i}\int\limits_{\partial Q}\varphi(s)\,\frac{dt_j}{t_j-t_{j0}}, \qquad z\in Q_e.$$

Here $Q_i = Q$, Q_e is an outer domain with respect to Q, $t_{j0} = t_j(s_0)$, and $\varphi(s)\in C^{0,\lambda}(\partial Q)$. From this it follows that

$$\sigma_{kj}(\boldsymbol{u}(x,\boldsymbol{f})n_j)_i(s_0) - \sigma_{kj}(\boldsymbol{u}(x,\boldsymbol{f})n_j)_e(s_0) = 2f_k(s_0), \qquad k=1,2.$$

Here $\sigma_{kj}(\boldsymbol{u}(x,\boldsymbol{f})n_j)_i(s_0)$ and $\sigma_{kj}(\boldsymbol{u}(x,\boldsymbol{f})n_j)_e(s_0)$ are the limiting values of the force vector with approach to the boundary from inside the domain and outside it, and $\boldsymbol{u}(x,\boldsymbol{f})$ is the simple layer potential defined below. Therefore, from boundary conditions (2) we obtain the following system of integral equations:

$$f_1(s_0) + \operatorname{Re}\frac{\mu_1 t_1'(s_0)}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(f_1+\mu_2 f_2)[t_1'(s)]^{-1}\,dt_1}{t_1-t_{10}}$$

$$-\operatorname{Re}\frac{\mu_2 t_2'(s_0)}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(f_1+\mu_1 f_2)[t_2'(s)]^{-1}\,dt_2}{t_2-t_{20}} = g_1(s_0); \tag{4}$$

$$f_2(s_0) - \operatorname{Re}\frac{t_1'(s_0)}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(f_1+\mu_2 f_2)[t_1'(s)]^{-1}\,dt_1}{t_1-t_{10}} + \operatorname{Re}\frac{t_2'(s_0)}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(f_1+\mu_1 f_2)[t_2'(s)]^{-1}\,dt_2}{t_2-t_{20}} = g_2(s_0). \tag{5}$$

Here $t_{k0} = x_1(s_0)+\mu_k x_2(s_0)$ $(k=1,2)$. System (4), (5) is conjugate after Fredholm to the system of equations

$$\psi_1(s_0) - \operatorname{Re}\frac{1}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(\psi_2-\mu_1\psi_1)\,dt_1}{t_1-t_{10}} + \operatorname{Re}\frac{1}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(\psi_2-\mu_2\psi_1)\,dt_2}{t_2-t_{20}} = h_1(s_0); \tag{6}$$

$$\psi_2(s_0) - \operatorname{Re}\frac{\mu_2}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(\psi_2-\mu_1\psi_1)\,dt_1}{t_1-t_{10}} + +\operatorname{Re}\frac{\mu_1}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}\frac{(\psi_2-\mu_2\psi_1)\,d\,t_2}{t_2-t_{20}} = h_2(s_0). \tag{7}$$

It is easy to verify by direct calculations that system (6), (7) has the eigenvector functions

$$\boldsymbol{w}_1 = (1,0), \quad \boldsymbol{w}_2 = (0,1), \quad \boldsymbol{w}_3 = (-x_2(s), x_1(s)).$$

The proof that all integrals appearing in system (6), (7) have at the utmost a weak singularity at the Lyapunov boundary is similar to that given in [12] and is therefore omitted. Consequently, for mutually conjugate systems of Eqs. (4) and (5) and Eqs. (6) and (7), all Fredholm theorems are valid. Thus, in order that Eqs. (4) and (5) be solvable, it is necessary and sufficient that their right sides be orthogonal to all solutions of the homogeneous equations (6) and (7), respectively, and vice versa.

Since system (4), (5) is intended to solve the first boundary-value problem, equilibrium conditions should be satisfied; i.e., the main vector and the resultant moment of the applied forces should vanish. Otherwise, the following equalities should hold:

$$\int\limits_{\partial Q} g_k(s)\,ds = 0, \quad k = 1,2, \qquad \int\limits_{\partial Q} (x_1(s)g_2(s) - x_2(s)g_1(s))\,ds = 0. \tag{8}$$

To verify this statement, we multiply Eqs. (4) and (5) into ds_0 and integrate the result over s_0 taking into account that

$$\int\limits_{\partial Q} \frac{t'_{k0}\,ds_0}{t_k - t_{k0}} = -\pi i.$$

Then, the left sides of Eqs. (4) and (5) vanish and we obtain the first two equalities (8). Similarly, we multiply Eq. (5) into $-x_2(s_0)$ and equation (4) into $x_1(s_0)$, change the order of integration, and integrate over s_0. As a result, we obtain the third equality of (8). Thus, equalities (8) are necessary in order that system (4), (5) have a solution.

The solution of the first boundary-value problem is given by the simple layer potential, which in this case is written as

$$\begin{aligned} u_1(x,y,\boldsymbol{f}) &= \mathrm{Re}\,\frac{p_1}{(\mu_1-\mu_2)\pi i}\int\limits_{\partial Q}(f_1+f_2\mu_2)\ln(z_1-t_1)\,ds - \mathrm{Re}\,\frac{p_2}{(\mu_1-\mu_2)\pi i}\int\limits_{\partial Q}(f_1+f_2\mu_1)\ln(z_2-t_2)\,ds,\\ u_2(x,y,\boldsymbol{f}) &= \mathrm{Re}\,\frac{q_1}{(\mu_1-\mu_2)\pi i}\int\limits_{\partial Q}(f_1+f_2\mu_2)\ln(z_1-t_1)\,ds - \mathrm{Re}\,\frac{q_2}{(\mu_1-\mu_2)\pi i}\int\limits_{\partial Q}(f_1+f_2\mu_1)\ln(z_2-t_2)\,ds \end{aligned} \tag{9}$$

(for definiteness, the main branch of the logarithm is chosen). Le us consider the properties of the simple layer potential in greater detail. In order that integrals (9) exist, it suffices that the densities f_1 and f_2 be continuous. In this case, u_1 and u_2 are continuous on the entire plane, but at infinity they grow in a logarithmic manner. The following lemma is valid.

Lemma 1. *For the simple layer potential with a continuous density $\boldsymbol{f} = (f_1, f_2)$ that satisfies the relation*

$$\int\limits_{\partial Q} f_i\,ds = 0, \qquad i = 1,2,$$

the following estimates are valid:

$$|u_i(x,f)| < \frac{c}{|x|}, \qquad \Big|\frac{\partial u_i}{\partial x_j}\Big| < \frac{c_1}{|x|^2}, \qquad i,j = 1,2, \quad |x| = \sqrt{x_1^2 + x_2^2}.$$

To prove the lemma, we consider the typical integral appearing in (9):

$$\int\limits_{\partial Q} f(s)\ln(z_k - t_k)\,ds.$$

Here $f(s)$ is any density. We write f as the derivative

$$f(s) = \frac{d}{ds}\Big(\int\limits_0^s f(s)\,ds\Big) = \frac{d}{ds}\tau(s).$$

Integration by parts yields the equality

$$\int\limits_{\partial Q} \frac{d\tau}{ds}\ln(z_k - t_k)\,ds = \tau(s)\ln(z_1 - t_1)\Big|_{\partial Q} - \int\limits_{\partial Q}\tau(s)\,\frac{dt_k}{t_k - z_k}. \tag{10}$$

The first term on the right side of formula (10) vanishes if $\tau(L) = 0$; the second term is a single-value Cauchy integral. The equality $\tau(L) = 0$ implies that the conditions of the lemma are satisfied. The remaining statements of the lemma are obvious. This implies that the functions $v_1(x_1,x_2)$ are $v_2(x_1,x_2)$ are single-valued.

We set $\boldsymbol{f} = (f_1, f_2)$ and denote the simple layer potential by

$$\boldsymbol{u}(x, \boldsymbol{f}) = (u_1(x, \boldsymbol{f}), u_2(x, \boldsymbol{f})), \qquad u_i(x, \boldsymbol{f}) = \sum_{j=1}^{2} \int\limits_{\partial Q} G_{ij} f_j(s)\, ds, \quad i = 1, 2.$$

Here G_{ij} $(i, j = 1, 2)$ are the cores of the integral operators in (9). We introduce the matrix $G(x) = (G_{ij}(x))$ $(i, j = 1, 2)$ and write the simple layer potential in vector form

$$\boldsymbol{u}(x, \boldsymbol{f}) = \int\limits_{\partial Q} G(x - y) \boldsymbol{f}\, ds.$$

Lemma 2. *The system of integral equations* (6) *and* (7) *has three and only three linearly independent solutions*

$$\boldsymbol{f}_1 = (1, 0), \qquad \boldsymbol{f}_2 = (0, 1), \qquad \boldsymbol{f}_3 = (-x_2(s), x_1(s)).$$

Indeed, if this statement were wrong, system (4), (5), which is conjugate to (6) and (7), would also have more than three linearly independent solutions $\boldsymbol{w}_k(s)$ $(k > 3)$. Each of them corresponds to a simple layer potential $\boldsymbol{v}(x, f_k)$, for which $\sigma_{ij}(\boldsymbol{v}(x, f_k) n_j) = 0$ on the boundary of the domain.

Let there exist one more solution $f_4(s)$ that is linearly independent of the previous ones; then, the expression

$$f(s) = f_4(s) - \sum_{j=1}^{3} c_j f_j(s)$$

[c_j $(j = 1, 2, 3)$ are arbitrary real constants] is also a solution of the homogeneous system of Eqs. (6) and (7). Let us compose the simple layer potentials

$$\boldsymbol{u}(x, f_4), \quad \boldsymbol{u}(x, f_j), \quad j = 1, 2, 3.$$

As solutions of homogeneous boundary-value problems, they are rigid-displacement vectors

$$\boldsymbol{u}(x) = \boldsymbol{u}(x, f_4) - \sum_{j=1}^{3} \boldsymbol{u}(x, f_j).$$

We choose the constants c_1, c_2, and c_3 such that the following conditions are satisfied:

$$\boldsymbol{u}(0) = 0, \qquad \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} = 0.$$

In this case, v^k satisfy the system

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = 0, \qquad i, j = 1, 2.$$

These conditions can be written as

$$\sum_{j=1}^{3} c_j \int\limits_{\partial Q} G(x_2) \varphi_j(s)\, ds = \int\limits_{\partial Q} G(x_2)\, ds,$$

$$\sum_{j=1}^{3} c_j \int\limits_{\partial Q} G_0(x_2) \varphi_j(s)\, ds = \int\limits_{\partial Q} G_0(x_2)\, ds,$$

where

$$G_0(x_2) = \frac{\partial G_2}{\partial x_2} - \frac{\partial G_1}{\partial x_1}.$$

The determinant of this system of equations is different from zero by virtue of the linear independence of w_j $(j = 1, 2, 3)$. Solving this system for c_j $(j = 1, 2, 3)$, we obtain $u(x_1, x_2) = 0$, where $(x_1, x_2) \in Q_i$. The continuity of the potential on the entire plane implies that $u(x) = 0$ on ∂Q. Since $u(x)$ obeys Lemma 1, at infinity we have

$$|u(x_1, x_2)| < c/\sqrt{x_1^2 + x_2^2}, \qquad |\sigma_{ij}(u(x))| < c_1/(x_1^2 + x_2^2).$$

Therefore, $u(x_1, x_2) = 0$, where $(x_1, x_2) \in Q_e$. But then, according to the Sokhotskii formula, the discontinuity of the force vector on the boundary is zero and $f_4 = 0$. We have an inconsistency. In the case where the force vector is specified on the boundary, we denote the solution of the internal boundary-value problem by (I, Q_i) and the solution of the external boundary-value problem by (I, Q_e). The following lemma is valid.

Lemma 3. *Problem (I, Q_i) has a single solution with accuracy up to the linear combination $c_1w_1 + c_2w_2 + c_3w_3$, where*

$$\boldsymbol{w}_1 = (0,1), \quad \boldsymbol{w}_2 = (1,0), \quad \boldsymbol{w}_3 = (-x_2, x_1),$$

and the functions w_k $(k = 1,2,3)$ constitute the full set of solutions of the system

$$\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i} = 0, \qquad i,k = 1,2.$$

The proof of Lemma 3 follows from the validity of the integral equality

$$\int\limits_{Q_i} \sigma_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\,dx = \int\limits_{\partial} Q\sigma_{ij}(\boldsymbol{v}) v_i \nu_j\,ds. \tag{11}$$

Here ν is an outward normal with respect to Q_i. For the functions $\boldsymbol{v}$ that have the properties

$$|\boldsymbol{v}| \le c, \qquad \left|\frac{\partial v_k}{\partial x_i}\right| \le \frac{c}{|x|^2} \quad (i,k = 1,2) \tag{12}$$

for large $|x|$ and satisfy the homogeneous system of elasticity equations in the unbounded domain Q_e, the following integral equality is valid:

$$\int\limits_{Q_e} \sigma_{ij}(\boldsymbol{v}) e_{ij}(\boldsymbol{v})\,dx = -\int\limits_{\partial} Q\sigma_{ij}(\boldsymbol{v}) v_i \nu_j\,ds. \tag{13}$$

Equality (13) follows from the integral equality (11) for a bounded domain. Indeed, we consider the domain located between the boundary ∂Q and a circle of large radius R with center lying inside Q_i. For this domain, equality (11) is valid. By virtue of (12), the integral over the outer circumference tends to zero as $R \to \infty$. Therefore, equality (13) follows from the vanishing of the integral over the circle of radius R and from the absolute convergence of the integral over Q_e. This implies the following lemma.

Lemma 4. *The solution of problem (I, Q_e) in the class of functions that possess property* (12) *is unique with accuracy up to a constant vector, and if $\boldsymbol{u} \to 0$ as $|x| \to \infty$, the solution of problem (I, Q_e) is unique.*

System (4), (5) can be modified so that it becomes solvable for any right side. Indeed, let us supplement Eq. (4) by the terms

$$-\operatorname{Re}\frac{\mu_1}{2\pi i(\mu_1 - \mu_2)}\frac{t_1'(s_0)}{t_1(s_0)}\int\limits_{\partial Q}(f_1 + \mu_2 f_2)\,ds + \operatorname{Re}\frac{\mu_2}{2\pi i(\mu_1 - \mu_2)}\frac{t_2'(s_0)}{t_2(s_0)}$$

$$\times \int\limits_{\partial Q}(f_1 + \mu_1 f_2)\,ds - \operatorname{Re}\frac{1}{4\pi i}\Big(-\mu_1\frac{\partial}{\partial s_0}\frac{1}{t_1(s_0)} - \mu_2\frac{\partial}{\partial s_0}\frac{1}{t_2(s_0)}\Big)M,$$

and Eq. (5) by the terms

$$-\operatorname{Re}\frac{1}{2\pi i(\mu_1 - \mu_2)}\frac{t_1'(s_0)}{t_1(s_0)}\int\limits_{\partial Q}(f_1 + \mu_2 f_2)\,ds + \operatorname{Re}\frac{1}{2\pi i(\mu_1 - \mu_2)}\frac{t_2'(s_0)}{t_2(s_0)}$$

$$\times \int\limits_{\partial Q}(f_1 + \mu_1 f_2)\,ds - \operatorname{Re}\frac{1}{4\pi i}\Big(\frac{\partial}{\partial s_0}\frac{1}{t_1(s_0)} + \frac{\partial}{\partial s_0}\frac{1}{t_2(s_0)}\Big)M,$$

where M is a real constant. Since

$$\int\limits_{\partial Q}\frac{dt_k(s_0)}{t_k(s_0)} = 2\pi i \qquad (k = 1,2)$$

(the coordinate origin is inside the domain), we have

$$\int\limits_{\partial Q} f_k(s)\,ds = \int\limits_{\partial Q} g_k(s)\,ds \qquad (k=1,2),$$

$$M = \int\limits_{\partial Q} (-x_2(s)g_1(s) + x_1(s)g_2(s))\,ds.$$

If the main vector and the resultant moment of the applied forces are equal to zero, the system of equations with these terms is equivalent to system (4), (5). In order that it be meaningful, it is necessary to set $g_k(s) \in C^{0,\lambda}(\partial Q)$ $(k=1,2)$. Then, $f_k(s) \in C^{0,\lambda}(\partial Q)$ $(k=1,2)$. It is not quite obvious that this is a system of second-order Fredholm equations. Equations (4) and (5) can be written as

$$f_1(s_0) + \operatorname{Re}\frac{t_1'(s_0)}{\pi i}\int\limits_{\partial Q} f_1(s)\frac{[t_1'(s)]^{-1}\,dt_1}{t_1 - t_{10}} + \operatorname{Re}\frac{\mu_2}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}(f_1+\mu_1 f_2)\Big(\frac{t_2'(s_0)}{t_2-t_{20}} - \frac{t_1'(s_0)}{t_1-t_{10}}\Big)\,ds = g_1(s_0); \qquad (14)$$

$$f_2(s_0) + \operatorname{Re}\frac{t_2'(s_0)}{\pi i}\int\limits_{\partial Q} f_2(s)\frac{(t_2'(s))^{-1}\,dt_1}{t_2 - t_{20}} + \operatorname{Re}\frac{1}{\pi i(\mu_1-\mu_2)}\int\limits_{\partial Q}(f_1+\mu_2 f_2)\Big(\frac{t_1'(s_0)}{t_1-t_{10}} - \frac{t_2'(s_0)}{t_2-t_{20}}\Big)\,ds = g_2(s_0). \qquad (15)$$

Thus, system (14), (15) is a system of second-order Fredholm equations.

The main result of the study is formulated as follows.

Theorem 1. *Let*

$$g_k(s) \in C^{0,\lambda}(\partial Q), \quad k=1,2, \qquad \partial Q \in C^{1,\lambda}(0,L), \quad 0<\lambda<1.$$

In order that system (14), (15) *have a solution in the class $C^{0,\lambda}(\partial Q)$, where $u_k(x) \in C^{1,\lambda}(Q)$ $(k=1,2)$, it is necessary and sufficient that the main vector and the resultant moment of the applied forces vanish. In this case, the solution of the boundary-value problem is $u_k(x) \in C^{1,\lambda}(\overline{Q})$ $(k=1,2)$.*

Proof. We seek a solution of the boundary-value problem in the form of the simple layer potential $\boldsymbol{u} = (u_1(x,\boldsymbol{f}),\ u_2(x,\boldsymbol{f}))$. Then, the density $\boldsymbol{f} = (f_1, f_2)$ satisfies system (4), (5). Since all Fredholm theorems are valid for the pair of conjugate integral equations (4), (5) and (6), (7), system (4), (5) is solvable if an only if the vector function $\boldsymbol{g} = (g_1, g_2)$ is orthogonal to all solutions of the homogeneous system of Eqs. (6) and (7). According to Lemma 2, the general solution of the homogeneous system of Eqs. (6) and (7) is the rigid-displacement vector $\boldsymbol{\varphi} = (\varphi_1, \varphi_2)$ ($\varphi_1 = -c_3x_2 + c_1$ and $\varphi_2 = c_3x_1 + c_2$, where c_1, c_2, and c_3 are arbitrary real constants). Equalities (8) imply that the solution of the homogeneous system (6), (7) is orthogonal to the vector function $\boldsymbol{g} = (g_1, g_2)$; therefore, they guarantee the existence of a solution of system (4), (5). The same equalities are necessary solvability conditions, as follows from the Betty formula:

$$0 = \int\limits_{\partial Q}\sigma_{ij}n_j\varphi_i\,ds = \int\limits_{\partial Q} g_i\varphi_i\,ds.$$

The relations obtained are equivalent to the equilibrium conditions. The smoothness of the solution is increased by increasing the smoothness of the boundary and the boundary data. The following theorem is valid.

Theorem 2. *Let*

$$g_k(s) \in C^{l,\lambda}(\partial Q), \quad k=1,2, \qquad \partial Q \in C^{l+1,\lambda}(0,L), \quad 0<\lambda<1, \quad l \ge 1.$$

Then, the solution of the boundary-value problem belongs to the class $C^{l+1,\lambda}(\overline{Q})$. This result follows from the smoothness properties of the Cauchy integral ([13, Theorem 8]).

2. We consider the solution for an isotropic material. In the case of isotropy,

$$a_{11} = a_{22} = 1/E, \quad a_{12} = -\nu/E, \quad a_{66} = 1/G, \quad a_{16} = a_{26} = 0,$$

where E is Young's modulus, G is the shear modulus, and ν is Poisson's constant. As $\mu_1, \mu_2 \to i$, the passage to the limit is performed. As a result, for an isotropic material we have the following system of equations:

$$f_1(s_0) + \operatorname{Re}\frac{t'(s_0)}{\pi i}\int\limits_{\partial Q}\frac{f_1(s)[t'(s)]^{-1}\,dt}{t-t_0} + \operatorname{Re}\frac{1}{2\pi i}\int\limits_{\partial Q}(f_1 + if_2)\,\frac{(\bar t - \bar t_0)\,dt - (t-t_0)\,d\bar t}{(t-t_0)^2} = g_1(s_0),$$

$$f_2(s_0) + \mathrm{Re}\,\frac{t'(s_0)}{\pi i}\int\limits_{\partial Q}\frac{f_2(s)[t'(s)]^{-1}\,dt}{t-t_0} + \mathrm{Re}\,\frac{i}{2\pi i}\int\limits_{\partial Q}(f_1+if_2)\,\frac{(\bar t-\bar t_0)\,dt-(t-t_0)\,d\bar t}{(t-t_0)^2} = g_2(s_0). \tag{16}$$

Here $t = x_1(s) + ix_2(s)$ and $t_0 = x_1(s_0) + ix_2(s_0)$. We bring system (16) to one (complex) equation for the complex(integrated) density $\omega(s) = f_1(s) + if_2(s)$. This system is equivalent to the system of equations in [6].

We denote by $(u_1^1(x), u_2^1(x))$ the displacement vector components for an isotropic material. Then,

$$u_1^1(x) = \mathrm{Re}\Big[\frac{2}{E}\,\frac{1}{\pi}\int\limits_{\partial Q} f_1(s)\ln(z-t)\,ds + \frac{1+\nu}{E}\,\frac{1}{\pi}\int\limits_{\partial Q} f_1(s)\,\frac{i(\eta-x_2)}{t-z}\,ds\Big]$$

$$+\,\mathrm{Re}\Big[\frac{1-\nu}{E}\,\frac{1}{\pi}\int\limits_{\partial Q} if_2(s)\ln(z-t)\,ds + \frac{1+\nu}{E}\,\frac{1}{\pi}\int\limits_{\partial Q} f_2(s)\,\frac{i(\eta-x_2)}{t-z}\,ds\Big],$$

$$u_2^1(x) = \mathrm{Re}\Big[\frac{1-\nu}{E}\,\frac{1}{\pi i}\int\limits_{\partial Q} f_1(s)\ln(z-t)\,ds - \frac{1+\nu}{E}\,\frac{1}{\pi i}\int\limits_{\partial Q} f_1(s)\,\frac{i(\eta-x_2)}{t-z}\,ds\Big]$$

$$+\,\mathrm{Re}\Big[\frac{2}{E}\,\frac{1}{\pi i}\int\limits_{\partial Q} if_2(s)\ln(z-t)\,ds - \frac{1+\nu}{E}\,\frac{1}{\pi i}\int\limits_{\partial Q} if_2(s)\,\frac{i(\eta-x_2)}{t-z}\,ds\Big],$$

where $z = x_1 + ix_2$ and $\eta = x_2(s)$.

We note that the above representation of displacements for an isotropic material has not been proposed previously.

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